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A Tiled Order Having Large Global Dimension

Willem S. Jansen

Compuware, 31440 Northwestern Highway, Farmington Hills, Michigan 48334-2564

Charles J. Odenthal

Mathematics Department, University of Toledo, Toledo, Ohio 43606-3390

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The finite global dimension of a tiled order over a discrete valuation ring is bounded by a function of the uniform dimension of the order, but the exact form of the function is unknown. Let N be the uniform dimension. A conjecture [T1] that the bound is $N-1$ was disproved [F2] by producing an example of global dimension N. We construct, for any even $N \geq 8$, a tiled order of uniform dimension N having global dimension $2N-8$.

Let D be a commutative discrete valuation ring with maximal ideal πD and field of quotients K. A classical D-order in the $N \times N$ matrix ring $M_N(K)$ is called a *tiled order* if it contains a set of N orthogonal idempotents. Note that the size of the matrix ring N equals the uniform dimension of the order.

Tarsy [T1] constructed a tiled order in $M_N(K)$, for $N \geq 2$, of global dimension $N-1$, and conjectured that no classical D-order in $M_N(K)$ has finite global dimension exceeding this bound. Much of the work on this conjecture has been confined to the study of tiled orders. While quite a bit easier to work with than arbitrary orders, the class of tiled orders is still large enough to have a rich and varied structure.

Among the results in line with the conjecture: Tarsy [T2] showed there is an upper bound on the finite finitistic dimensions (and so on the finite global dimensions) of tiled orders in $M_N(K)$; Jategaonkar [J1] proved

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Tarsy's conjecture for triangular tiled orders and [J2] for tiled orders with $N = 2, 3, 4$; Kirkman and Kuzmanovich [KK] proved Tarsy's conjecture for tiled orders containing the radical of a maximal over order; Fujita [F2] proved Tarsy's conjecture for tiled orders with $N = 5$.

However, Fujita [F2, Example 2.5] constructed a counterexample to Tarsy's conjecture: for any $N \geq 6$, a tiled order in $M_N(K)$ having global dimension N. Also, Rump [R, Example 3] has recently constructed a tiled order in $M_8(K)$ having global dimension 9. So the question remains: What is the bound on the global dimension of tiled orders in $M_N(K)$? In this paper we modify the example of Fujita cited above to obtain a tiled order in $M_N(K)$ (for even $N \geq 8$) with global dimension $2N - 8$.

Some of the constructions used to prove this result have formulas that make sense only for "large" N. To be precise: the definition of the "X" modules in Section 4 and the definition of the "T" modules in Section 6 require $N \geq 12$, while the proofs described for the results in Sections 4–6 require $N \geq 26$. However, the tiled order Λ_N described in Section 2 has global dimension $2N - 8$ for every even $N \geq 8$. The verification of this result for $N = 8, 10, \ldots, 24$ was done independently with the aid of a computer program written by the authors.

Throughout this paper, all modules will be finitely generated right modules.

1. TILED ORDERS AND LINK DIAGRAMS

Throughout, N will be an integer, and e_{ij}, $1 \leq i, j \leq N$, will be the canonical set of matrix units in $M_N(K)$. All our tiled orders Γ will contain the orthogonal idempotents $e_{11}, \ldots, e_{NN}$, so

$$\Gamma = (\pi^{\gamma_{ij}} D) \subset M_N(K).$$

The *exponent matrix* $(\gamma_{ij})_{1 \leq i, j \leq N}$ completely characterizes Γ, and we will identify Γ and its exponent matrix. Note that any such matrix of integers corresponds to some tiled order provided $\gamma_{ii} = 0$ and $\gamma_{ij} \leq \gamma_{ik} + \gamma_{kj}$ for every $1 \leq i, j, k \leq N$. Much of the information contained in the exponent matrix is redundant. In fact, this matrix is completely determined by those entries γ_{ij} (with $i \neq j$) satisfying

$$\gamma_{ij} < \gamma_{ik} + \gamma_{kj} \qquad \text{for all } k \neq i, j. \tag{1.1}$$

We use this inequality to define the *link diagram* of Γ, a directed graph with weighted edges.

The vertex set for the link diagram is $1, \ldots, N$. If (1.1) holds for vertices $i \neq j$ there is said to be a link from j to i of weight γ_{ij} (notation: $j \to_{\gamma_{ij}} i$). We will write $j \to i$ if the weight is irrelevant. If $1 < \gamma_{ik} + \gamma_{ki}$ for all $k \neq i$ there is said to be a self-link $i \to i$. All self-links have weight 1.

It can then be seen that Γ is completely determined by its link diagram: γ_{ij} is just the minimum among all weights of (directed) paths from j to i. The next lemma is easily verified.

LEMMA 1.1. *An arbitrary directed graph with weighted edges is the link diagram of some tiled order provided*:

(1) *the graph is directed-path-connected*;

(2) *the weight of an edge $j \to i$ is strictly smaller than the total weight of any other path from j to i*;

(3) *every vertex lies on a cycle of total weight* 1;

(4) *every cycle has nonnegative total weight*.

Note that replacing (4) *with*

(4′) *every cycle has positive total weight*

gives a characterization of the link diagrams of basic tiled orders.

We now introduce some notation that will be used throughout the paper. Let $J = J(\Gamma)$ be the Jacobson radical of Γ. Let $P(i) = e_{ii}\Gamma$, the ith indecomposable projective, i.e. the ith row of Γ, and let $J(i) = e_{ii}J$, the radical of $P(i)$. Let $S_i = P(i)/J(i)$, the ith simple module, and let M_i be its annihilator. Note that M_i is the maximal ideal of Γ obtained by replacing the D in the ith diagonal position of Γ with πD. Also let $J^m(i) = e_{ii}J^m$.

The following lemma gives additional characterizations of links. See [F1, Lemma 4.1] for a proof.

LEMMA 1.2. *The following are equivalent for a pair of indices* $1 \leq i, j \leq N$:

(1) *there is a link $j \to i$*;

(2) *$P(j)$ is a summand of the projective cover of $J(i)$*;

(3) *S_j is a summand of $J(i)/J^2(i)$*;

(4) *there is a link $M_j \rightsquigarrow M_i$ of maximal ideals, i.e. there is a proper inclusion $M_iM_j \subset M_i \cap M_j$*.

We will make heavy use of the fact that any torsion-free uniform Γ module U can be embedded in the unique simple $M_N(K)$ module $[K \cdots K]$. Pick such an embedding, then

$$U = [\pi^{u_1}D \cdots \pi^{u_N}D] \subset [K \cdots K].$$

Note that a different embedding of U in $[K \cdots K]$ would necessarily have an image of the form $\pi^t U = [\pi^{t+u_1}D, \dots, \pi^{t+u_N}D]$, and this has exponent vector $[t + u_1, \dots, t + u_N]$. So U determines its exponent vector up to a constant shift, and the exponent vector determines U up to isomorphism. Also observe that any such vector of integers $[u_1, \dots, u_N]$ is the exponent vector of some torsion-free uniform Γ module provided $u_i + \gamma_{ij} \geq u_j$ for every $1 \leq i, j \leq N$. Accordingly, we identify any such U with the *exponent vector* $[u_1, \dots, u_N]$ corresponding to some embedding. In particular, for the projective modules $P(i)$ we choose the canonical embedding and identify $P(i)$ with the ith row of the exponent matrix of Γ:

$$P(i) = [\gamma_{i1}, \dots, \gamma_{iN}].$$

For $m > 0$ let ρ_{ij}^m be the entry in row i and column j of the exponent matrix for J^m. In particular, $\rho_{ij}^1 = \gamma_{ij}$ if $i \neq j$ and $\rho_{ii}^1 = 1$. We make the canonical identification of $J^m(i)$ with the ith row of the exponent matrix of J^m:

$$J^m(i) = [\rho_{i1}^m, \dots, \rho_{iN}^m].$$

Using the above identifications, we now record some easy facts about projective covers, syzygies, and quotients of torsion-free uniform Γ modules. First an easy analog of Lemma 1.2.

LEMMA 1.3. *For $U = [u_1, \dots, u_N]$ the following are equivalent*:

(1) $u_i + \gamma_{ij} > u_j$ *for all* $i \neq j$;

(2) $P(j)$ *is a summand of the projective cover of* U;

(3) S_j *is a summand of* U/UJ.

In the above situation $P(j)$ occurs in the projective cover of U with multiplicity 1 *and the covering map is given by the inclusion of $\pi^{u_j}P(j)$ in $[K, \dots, K]$.*

In particular, if $U = J(i)$ the covering map is

$$\begin{aligned} P(j) = \left[\rho_{j1}^1, \dots, \rho_{jN}^1\right] &\to \left[\gamma_{ij} + \rho_{j1}^1, \dots, \gamma_{ij} + \rho_{jN}^1\right] \\ &\subseteq \left[\rho_{i1}^1, \dots, \rho_{iN}^1\right] = J(i). \end{aligned}$$

Now consider a short exact sequence

$$0 \to V \to U \to T \to 0,$$

where $V = [v_1, \dots, v_N]$ and $U = [u_1, \dots, u_N]$ are torsion-free uniform modules, and the map from V to U is inclusion.

LEMMA 1.4. *With the above notation $v_j \geq u_j$ for all $1 \leq j \leq N$, and*

(1) *the simple module S_j is a composition factor of T with multiplicity $t > 0 \Leftrightarrow v_j - u_j = t$;*

(2) *no composition factor of T occurs more than once in any given Loewy layer TJ^{m-1}/TJ^m.*

Proof. A composition series for T corresponds to a series $V = W_0 \subset W_1 \cdots \subset W_s = U$ where $W_k/W_{k-1} \cong S_j$ means (identifying the W's with their exponent vectors) that W_k and W_{k-1} differ by 1 in column j and are equal in all other columns. Part (1) now follows.

Suppose $W = [w_1, \dots, w_N]$ is a torsion-free uniform module and write $WJ = [w'_1, \dots, w'_N]$. It is easy to see that $w_j + 1 \geq w'_j \geq w_j$ for all $1 \leq j \leq N$. So no S_j occurs with multiplicity more than 1 as a summand of W/WJ. Now, taking $W = UJ^{m-1}$, (2) follows since $TJ^{m-1}/TJ^m = (UJ^{m-1} + V)/(UJ^m + V)$ is a quotient of W/WJ. ∎

We return to the consideration of $J^m(i)$. It is easy to see that $\rho_{ij}^2 = \min_k\{\rho_{ik}^1 + \rho_{kj}^1\}$. Then by induction we have

$$\rho_{ij}^m = \min_{k_1, \dots, k_{m-1}} \left\{ \rho_{ik_1}^1 + \cdots + \rho_{k_{m-1}j}^1 \right\}.$$

This formula leads directly to the following lemma. The *length* of a path is just the number of links in the path.

LEMMA 1.5. *The entry $\rho_{ij}^m = \rho_{ij}^1$ if and only if the link diagram has a path from j to i having weight ρ_{ij}^1 and length at least m.*

COROLLARY 1.6. *The following are equivalent for an integer $m > 0$:*

(1) *the simple module S_j is a summand of $J^m(i)/J^{m+1}(i)$ but not of $J^k(i)/J^{k+1}(i)$ for any $0 < k < m$;*

(2) *the exponent m is minimal with $\rho_{ij}^{m+1} > \rho_{ij}^1$;*

(3) *there is a path in the link diagram from j to i having weight ρ_{ij}^1 and length precisely m, and all longer paths have greater weights.*

In the situation of the corollary it is easy to see that $\rho_{ij}^{m+1} = 1 + \rho_{ij}^1$.

We will need the special case of Lemma 1.4 when $U = J(i)$ for some i, that is, we have the short exact sequence

$$0 \to V \to J(i) \to T \to 0$$

for some nonzero V a submodule of $J(i)$. Combining Lemma 1.4 with Corollary 1.6 gives the following.

COROLLARY 1.7. *With the above notation*

(1) *the simple module* S_j *is a composition factor of* T *with multiplicity* $t > 0 \Leftrightarrow v_j - \rho_{ij}^1 = t$;

(2) *no composition factor of* T *occurs more than once in any given Loewy layer* TJ^{m-1}/TJ^m;

(3) *if the simple module* S_j *occurs as a composition factor of* T, *then* S_j *occurs as a summand of the Loewy layer* TJ^{m-1}/TJ^m *and no previous* (*smaller exponent*) *Loewy layer for the minimal m such that* $\rho_{ij}^{m+1} > \rho_{ij}^1$.

In (3) recall that $\rho_{ij}^1 = \gamma_{ij}$ for $i \neq j$, and that $\rho_{ii}^1 = 1$.

Proof. Only (3) requires comment. Here just observe that if S_j is a summand of TJ^{m-1}/TJ^m and no previous Loewy layer, then S_j is a summand of $J^m(i)/J^{m+1}(i)$ and no previous Loewy layer. Now use Corollary 1.6. ∎

If the projective cover of a module has large uniform dimension, it can be an onerous task to decompose the syzygy into a direct sum of indecomposable modules. Unless this decomposition is performed, the computation of the next syzygy will be needlessly complicated. However, if the projective cover of a uniform module has uniform dimension 2 or 3, the syzygy can be easily decomposed into a direct sum of indecomposable modules. The following lemmas are easily verified. For any module X let ΩX denote the syzygy of X.

LEMMA 1.8. *If* $U = [u_1, \dots, u_n]$ *has projective cover* $P(h) \oplus P(k)$, *then the syzygy*

$$\begin{aligned} \Omega U &= \pi^{u_h} P(h) \cap \pi^{u_k} P(k) \\ &= \left[\dots, \max\{u_h + \gamma_{hj}, u_k + \gamma_{kj}\}, \dots \right]. \end{aligned}$$

LEMMA 1.9. *If* $U = [u_1, \dots, u_n]$ *has projective cover* $P(g) \oplus P(h) \oplus P(k)$, *then the syzygy*

$$\Omega U = W/V,$$

where

$$\begin{aligned} W &= \left[\pi^{u_g} P(g) \cap \pi^{u_h} P(h)\right] \oplus \left[\pi^{u_h} P(h) \cap \pi^{u_k} P(k)\right] \\ &\qquad \oplus \left[\pi^{u_g} P(g) \cap \pi^{u_k} P(k)\right], \\ V &= \left[\pi^{u_g} P(g) \cap \pi^{u_h} P(h) \cap \pi^{u_k} P(k)\right]. \end{aligned}$$

LEMMA 1.10. *If $U = [u_1, \dots, u_n]$ has projective cover $P(g) \oplus P(h) \oplus P(k)$ and $u_g + \gamma_{gj} \le \max\{u_h + \gamma_{hj}, u_k + \gamma_{kj}\}$ for all j then the syzygy*

$$\begin{aligned}\Omega U &= \left[\pi^{u_g}P(g) \cap \pi^{u_h}P(h)\right] \oplus \left[\pi^{u_g}P(g) \cap \pi^{u_k}P(k)\right] \\ &= \left[\dots, \max\{u_g + \gamma_{gj}, u_h + \gamma_{hj}\}, \dots\right] \\ &\qquad \oplus \left[\dots, \max\{u_g + \gamma_{gj}, u_k + \gamma_{kj}\}, \dots\right].\end{aligned}$$

2. THE EXAMPLE DESCRIBED

From now on N will be even with $n = N/2$, and Λ_N will be the example mentioned in the introduction with gl.dim$(\Lambda_N) = 2N - 8$. In this section only we will indicate explicitly the size of the matrix ring $M_N(K)$ containing a tiled order, e.g., Λ_N. In this section we define Λ_N as well as Δ_N, a naturally associated equivalent order that is used to simplify some of the calculations.

We first record a few facts about Λ_N and Δ_N that can be easily verified once the definitions of these two orders have been given. Both Λ_N and Δ_N are basic tiled orders. The link graph of Δ_N has "self-links" $k \to_1 k$ for each vertex $k = 1, \dots, N$. Consequently, by [J2, Lemma 1.7] the global dimension of Δ_N is infinite. We conjecture that the left and right finitistic dimensions of Δ_N are bounded above by N. This has been verified for $N \le 12$. Also, both Δ_N and Λ_N are of wild representation type since their posets contain the critical poset $(1, 1, 1, 1, 1)$ of Nazarova; see [S] for details. Let μ be the product of disjoint transpositions $(12)(34) \cdots (N-1N)$. Then the map on $M_N(K)$ defined by taking the permutation of matrix units $e_{ij} \mapsto e_{\mu(j)\mu(i)}$ and extending linearly is an involution of $M_N(K)$. This involution restricts to involutions of Δ_N and Λ_N.

We now describe the orders Λ_N and Δ_N by (1) specifying the weighted link graph of Δ_N, (2) defining the exponent matrix of Λ_N in terms of that of Δ_N, and (3) showing the link graph of Λ_N has the same underlying (unweighted) directed graph as the link graph of Δ_N except that Λ_N has no self-links. This common underlying directed graph is given in the next definition.

DEFINITION 2.1. Let G be a directed graph on N vertices with $9n - 13$ links (edges) in the following positions:

(i) links between odd vertices,

$2k - 1 \to 2k + 1$ for $k = 1, \dots, n - 1$,

$2k + 1 \to 2k - 1$ for $k = 1, \dots, n - 1$,

$1 \to 7$;

(ii) links between even vertices,

$2k \to 2k+2$ for $k = 1, \dots, n-1$,

$2k+2 \to 2k$ for $k = 1, \dots, n-1$,

$8 \to 2$;

(iii) links with an odd source and even sink,

$2k-1 \to 2k-4$ for $k = 3, \dots, n$,

$2k-1 \to 2k+4$ for $k = 1, \dots, n-2$,

$1 \to 2$;

(iv) links with an even source and odd sink,

$2k-8 \to 2k-1$ for $k = 5, \dots, n$,

$2k \to 2k-1$ for $k = 1, \dots, n$,

$2k+8 \to 2k-1$ for $k = 1, \dots, n-4$.

Some links of type (iv) do not exist for $N = 8$.

We can now define Δ_N.

DEFINITION 2.2. The link graph for Δ_N has a link of weight 1 for each link in G as well as a self-link $k \to_1 k$ for each vertex $k = 1, \dots, N$.

An appeal to Lemma 1.1 shows that the above does actually define a tiled order. As a consequence of this definition δ_{ij}, the entry of the exponent matrix for Δ_N, is simply the minimal length of a path from vertex j to vertex i in G. In preparation to defining Λ_N let ν be the permutation of $1, \dots, N$ given by

$$\nu(k) = \begin{cases} n + \dfrac{1-k}{2} & \text{for } k \text{ odd}, \\[2ex] n + \dfrac{k}{2} & \text{for } k \text{ even}. \end{cases} \tag{2.1}$$

DEFINITION 2.3. The exponent matrix for Λ_N is given by

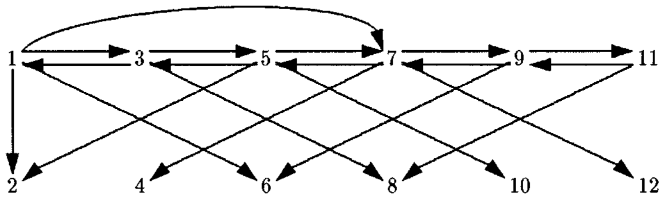
$$\lambda_{ij} = \tfrac{1}{2}\left[\nu(j) - \nu(i) + \delta_{ij}\right]. \tag{2.2}$$

Note that

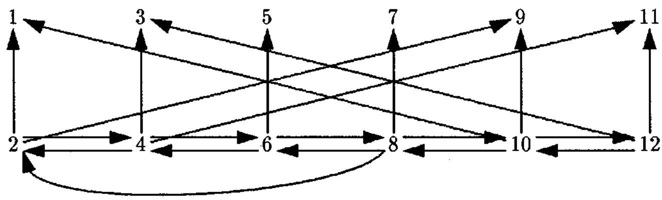
$$\lambda_{ik} + \lambda_{kj} = \tfrac{1}{2}\left[\nu(j) - \nu(i) + \delta_{ik} + \delta_{kj}\right] \geq \tfrac{1}{2}\left[\nu(j) - \nu(i) + \delta_{ij}\right] = \lambda_{ij}$$

with strict inequality if and only if $\delta_{ik} + \delta_{kj} > \delta_{ij}$. So this definition does give us a ring; and, for $i \neq j$, Λ_N has a link $j \to i$ precisely when Δ_N has one. Also note that, for any $1 \leq i \leq N-2$, Λ_N has links $i \to_1 i+2 \to_0 i$, and, for any $3 \leq i \leq N$, Λ_N has links $i \to_0 i-2 \to_1 i$. So every vertex i of Λ_N lies on at least one cycle of total weight 1. Since self-links must have weight 1, this implies that Λ_N has no self-links.

The directed graph G_{12} and exponent matrix for Λ_{12} are given below.



Links for $\mathcal{G}_{12}$ with odd sources



Links for $\mathcal{G}_{12}$ with even sources

$$\begin{bmatrix}
0 & 1 & 0 & 2 & 0 & 3 & 0 & 3 & 0 & 3 & -1 & 4 \\
0 & 0 & 0 & 1 & -1 & 2 & -1 & 2 & -1 & 3 & -2 & 4 \\
1 & 2 & 0 & 2 & 0 & 3 & 0 & 4 & 0 & 4 & 0 & 4 \\
0 & 0 & 0 & 0 & -1 & 1 & -2 & 2 & -2 & 3 & -2 & 4 \\
2 & 3 & 1 & 3 & 0 & 3 & 0 & 4 & 0 & 5 & 0 & 5 \\
-1 & 0 & -1 & 0 & -1 & 0 & -2 & 1 & -3 & 2 & -3 & 3 \\
2 & 3 & 2 & 4 & 1 & 4 & 0 & 4 & 0 & 5 & 0 & 6 \\
-1 & 0 & -2 & 0 & -2 & 0 & -2 & 0 & -3 & 1 & -4 & 2 \\
3 & 3 & 3 & 4 & 2 & 5 & 1 & 5 & 0 & 5 & 0 & 6 \\
-1 & 0 & -2 & 0 & -3 & 0 & -3 & 0 & -3 & 0 & -4 & 1 \\
4 & 4 & 4 & 4 & 3 & 5 & 2 & 6 & 1 & 6 & 0 & 6 \\
-2 & -1 & -2 & 0 & -3 & 0 & -4 & 0 & -4 & 0 & -4 & 0
\end{bmatrix}$$

Exponent matrix for Λ_{12}

We now proceed to derive recursive relations for the δ_{ij} that, by repeatedly passing from N to $N-2$, will allow us to verify relations for general N using the exponent matrix Δ_{24}.

In the transition from Λ_{N-2} to Λ_N there are nine new links: $N-1 \to N-4$, $N-1 \to N-3$, $N-8 \to N-1$, $N-3 \to N-1$, $N \to N-1$, $N \to N-2$, $N \to N-9$, $N-2 \to N$, and $N-5 \to N$. We proceed to show, letting $\epsilon = e_{11} + \cdots + e_{N-2,N-2}$, that Δ_{N-2} may be identified with $\epsilon\Delta_N\epsilon$ provided $N \geq 10$. Similarly Λ_{N-2} may be identified with $\epsilon\Lambda_N\epsilon$.

LEMMA 2.4. *Given two vertices j and i in G with* $\max\{i, j\} \geq 10$, *there is a minimal length path from j to i such that all intermediate vertices $k <$* $\max\{i, j\}$.

Proof. It is enough to consider paths $l \to k \to m$ of length 2 with $k > \max\{l, m, 10\}$ and show there is a path $l \to p \to m$ with $p < k$.

The proof now consists of a routine check (using Definition 2.1) of the various possibilities for $l \to k \to m$ with $k > \max\{l, m, 10\}$. For example, if k is odd, the path $k - 7 \to k \to k - 2$ can be replaced with $k - 7 \to k - 9 \to k - 2$. The other cases are similar. ∎

As a consequence of this lemma, the entry δ_{ij} is independent of N, provided only that $N \geq \max\{i, j\} \geq 10$. It can also be shown that the entries δ_{ij} where $\max\{i, j\} < 10$ are independent of N (provided $N \geq 10$): Using the lemma once again, it suffices to show that for every path $j \to k \to i$ with $i, j < 10$ and $k \geq 10$ there is either a link $j \to i$ or a path $j \to m \to i$ with $m < 10$. This can be done with a routine inspection of G_{16}. Since the difference $\nu(j) - \nu(i)$ is also independent of N, so is λ_{ij} by (2.2).

PROPOSITION 2.5. *We can (and do) make the following identifications*:

$$\Delta_{N-2} = \epsilon\Delta_N\epsilon \qquad \text{and} \qquad \Lambda_{N-2} = \epsilon\Lambda_N\epsilon,$$

where $\epsilon = e_{11} + \cdots + e_{N-2,N-2}$. We also form the directed unions of exponent matrices

$$\Delta_\infty = \bigcup_N \Delta_N \qquad \text{and} \qquad \Lambda_\infty = \bigcup_N \Lambda_N.$$

Note that, while these infinite matrices are matrices of exponents, they are not the exponent matrices of any ring of interest. However, we will abuse notation and refer to them as exponent matrices. We proceed to establish some recurrence relations that hold for the δ_{ij} in the exponent matrix Δ_∞. These relations will be used heavily in later sections. First we need a preliminary lemma.

We define the infinite directed graph:

$$G_\infty := \bigcup_N G_N.$$

This is just the (unweighted) link graph corresponding to the exponent matrix Δ_∞, i.e. for every pair of vertices i and j in G_∞, the shortest path from j to i has length δ_{ij}.

LEMMA 2.6. *Given two vertices j and i in G_∞ there is a minimal length path from j to i such that all intermediate vertices $k > \min\{i, j\}$.*

Proof. It is enough to consider paths $l \to k \to m$ of length 2 with $k < \min\{l, m\}$ and show there is a path $l \to p \to m$ with $p > k$. The proof now consists of a routine check (using Definition 2.1) of the various possibilities as in the proof of Lemma 2.4, the only difference being there are a few more cases to consider since $k = 1$ and $k = 2$ must be considered separately from the case of general odd or even k. ∎

RECURSION RELATIONS 2.7. *Given indices i, j, we have*

(i) *Shift* 2 *Relation*:

$$\delta_{ij} = \delta_{i-2,j-2} \qquad \text{if } i - 2 \neq 2 \text{ and } j - 2 \neq 1.$$

(ii) *Shift* 12 *Relation*:

$$\delta_{ij} = \begin{cases} 2 + \delta_{i-12,j} & \text{if } i \geq 13 \text{ and } j \in \{1, 2, 3\}, \\ 2 + \delta_{i,j-12} & \text{if } j \geq i + 12. \end{cases}$$

A paraphrase may be in order. The above relations should be thought of as statements about the infinite exponent matrix Δ_∞.

The Shift 2 Relation says that, provided one does not end up in row 2 or column 1, moving two steps up and to the left along an off diagonal does not change the value of δ.

The Shift 12 Relation says that moving 12 steps towards the main diagonal in a horizontal or vertical direction decreases the value of δ by 2. This motion is subject to a few restrictions: it must stay within the matrix (positive indices); the vertical movement must be in columns 1, 2, or 3; the horizontal movement must be from right to left and should not cross the main diagonal.

The proofs of these recurrence relations will occupy the remainder of this section.

Proof of Shift 2 *Relation.* Define another infinite directed graph G'_∞ by removing the three exceptional edges $1 \to 7$, $1 \to 2$, and $8 \to 2$ from G_∞. Let δ'_{ij} be the minimal length of a path from vertex j to vertex i in G'_∞. By Lemma 2.6, for any vertices s and t, a path from t to s of minimal length in G_∞ can be found that only passes through vertices $k \geq \min\{s, t\}$. In particular, if $s \neq 2$ and $t \neq 1$, then none of the three exceptional edges need be used, and the above path lies in G'_∞. Consequently, $\delta_{s,t} = \delta'_{s,t}$ provided $s \neq 2$ and $t \neq 1$. Since we are assuming $i - 2 \neq 2$ and $j - 2 \neq 1$ (and $i \geq 3$, $j \geq 4$), we have $\delta_{i-2,j-2} = \delta'_{i-2,j-2}$ and $\delta_{ij} = \delta'_{ij}$. It now suffices to show that $\delta'_{ij} = \delta'_{i-2,j-2}$.

Consider the embedding τ: $G'_\infty \to G'_\infty$ given by $k \mapsto k+2$ for vertices k. If p is a minimal length path from $j-2$ to $i-2$ in G'_∞, then $\tau(p)$ is a path in G'_∞ from j to i, so $\delta'_{ij} \le \delta'_{i-2,j-2}$.

Conversely, since $i \ge 3$ and $j \ge 4$ Lemma 2.6 guarantees that there is a path p of minimal length from j to i in G'_∞ with all intermediate vertices $k \ge \min\{i, j\}$. Then $p = \tau(q)$ for some paths q from $j-2$ to $i-2$ in G'_∞. So $\delta'_{ij} \ge \delta'_{i-2,j-2}$, and the Shift 2 recurrence relation is established. ∎

Proof of Shift 12 *Relation*. We first consider the equation $\delta_{ij} = \delta_{i-12,j} + 2$ for $i \ge 13$ and $j \in \{1,2,3\}$. We can see this equation is satisfied for $24 \ge i \ge 13$ by inspecting the exponent matrix for Δ_{24} found at the end of Section 3. So we assume $i \ge 25$.

Since $\delta_{ij} \le \delta_{i,i-12} + \delta_{i-12,j} \le 2 + \delta_{i-12,j}$ it suffices to prove the reverse inequality. We induct on δ_{ij}, the minimal length of a path from j to i in G_∞.

Pick a vertex k subject to $\delta_{ij} = 1 + \delta_{ik}$ where $j \to k$ is a link. Then there is a minimal length path

$$j \to k \to \cdots \to i$$

and $\delta_{ik} < \delta_{ij}$. Since $j \in \{1,2,3\}$ and $j \to k$ we see from Definition 2.1 that $1 \le k \le 9$. We accordingly apply the Shift 2 Relation t times $(t \le 3)$ to obtain $\delta_{ik} = \delta_{i-2t,k-2t}$ with $i - 2t \ge 19$ and $k - 2t \in \{1,2,3\}$. We have

$$\begin{aligned}
\delta_{ij} &= \delta_{ik} + \delta_{kj} \\
&= \delta_{i-2t,k-2t} + \delta_{kj} \qquad \text{by the Shift 2 Relation} \\
&= (2 + \delta_{i-2t-12,k-2t}) + \delta_{kj} \qquad \text{by induction} \\
&= 2 + \delta_{i-12,k} + \delta_{kj} \qquad \text{by the Shift 2 Relation} \\
&\ge 2 + \delta_{i-12,j}.
\end{aligned}$$

Now consider the equation $\delta_{ij} = 2 + \delta_{i,j-12}$ for $j \ge i + 12$. After repeatedly applying the Shift 2 Relation if necessary we may assume $i \in \{1,2,4\}$. Another inspection of the exponent matrix for Δ_{24} shows we may also assume $j \ge 25$. The argument is now analogous to that given for the other equation and we omit it. ∎

3. SKETCH OF PROOF DETAILS

In Sections 4 and 5 we define a number of torsion-free uniform Λ modules by giving their projective covers. We then compute the syzygies of these uniform modules. In Section 6 we define a number of torsion Λ modules by presenting them as quotients of torsion-free uniform modules. We then compute their composition factors.

The present section is devoted to a sketch of the technique used to prove these results from Sections 4–6. All the proofs use the same reduction method which we will illustrate with an example. But, first we give a brief outline of the argument.

The proofs of all the results proceed along the same mechanical lines: we interpret each statement to be proved as equations satisfied by the entries λ_{ij} of the exponent matrix of Λ, and then use (2.2) to rewrite these equations as equivalent equations in the entries δ_{ij} of the exponent matrix Δ_∞.

Then, given one of these equations in the δ_{ij}, we repeatedly apply the Shift 2 and Shift 12 Relations until we obtain an equivalent equation, all of whose entries δ_{ij} have $i, j \leq 24$. This equivalent equation is then verified by inspecting the Δ_{24} matrix (reproduced at the end of this section).

The above reduction is possible because, in every equation we need to consider, all the δ_{ij} lie in a single column with row indices no more than nine positions apart. The algorithm is to first apply the Shift 2 Relation, reducing all indices in an equation by 2, as many times as possible. We are (only) obliged to stop when the equation has column index $t \in \{1, 2, 3\}$, or the smallest row index $s \in \{1, 2, 4\}$. Next, we repeatedly apply the Shift 12 Relation. If $t \in \{1, 2, 3\}$ we reduce the row indices by 12 provided the smallest row index is at least 13. If $s \in \{1, 2, 4\}$ we reduce the column index by 12 provided it is at least 12 more than the largest row index.

Now, for an example of the technique, consider the relation $\Omega X_k = X_{k-1}$ from Proposition 4.2 for $2 \leq k \leq n - 2$. Using the definitions of X_k and X_{k-1} (Definition 4.1), we can express this relation as

$$\pi^{-k}P(2k-1) \cap P(2k+2) = \pi^{-(k-1)}P(2(k-1)-1) + P(2(k-1)+2)$$

and Lemma 1.8 allows us to interpret this as the equations

$$\max\{-k + \lambda_{2k-1,j}, \lambda_{2k+2,j}\} = \min\{1 - k + \lambda_{2k-3,j}, \lambda_{2k,j}\}$$

for $j = 1, \ldots, N$. Using (2.2), this becomes

$$\begin{aligned}\max\Big\{&-k + \tfrac{1}{2}\big(\nu(j) - \nu(2k-1) + \delta_{2k-1,j}\big),\\ &\tfrac{1}{2}\big(\nu(j) - \nu(2k+2) + \delta_{2k+2,j}\big)\Big\}\\ = \min\Big\{&1 - k + \tfrac{1}{2}\big(\nu(j) - \nu(2k-3) + \delta_{2k-3,j}\big),\\ &\tfrac{1}{2}\big(\nu(j) - \nu(2k) + \delta_{2k,j}\big)\Big\}.\end{aligned}$$

Now multiplying by 2 and using the definition of ν [Eq. (2.1)] gives us

$$\begin{aligned}&\max\{-2k - (n-k+1) + \delta_{2k-1,j}, -(n+k+1) + \delta_{2k+2,j}\}\\ &\quad = \min\{2 - 2k - (n-k+2) + \delta_{2k-3,j}, -(n+k) + \delta_{2k,j}\}.\end{aligned}$$

This in turn becomes

$$-1 + \max\left\{\delta_{2k-1,j}, \delta_{2k+2,j}\right\} = \min\left\{\delta_{2k-3,j}, \delta_{2k,j}\right\}.$$

Notice that the smallest row index is odd for these particular equations.

Take one of these equations with $j \leq 2k - 3$ (the smallest row index). The Shift 2 Relation can be applied repeatedly (since the smallest row index will always be odd and at least as large as the column index) until the equation has column index $t \in \{1, 2, 3\}$. Then the Shift 12 Relation can be applied until the equation has smallest (and odd) row index $s \leq 11$. We obtain an equation of the form $-1 + \max\{\delta_{s+2,t}, \delta_{s+5,t}\} = \min\{\delta_{s,t}, \delta_{s+3,t}\}$ for some $s \in \{1, 3, 5, 7, 9, 11\}$ and $t \in \{1, 2, 3\}$. All of these 18 equations do in fact hold.

Now take one of the original equations with $j > 2k - 3$. In this case the smallest row index is still odd, but now smaller than the column index. So, the Shift 2 Relation can be applied until the equation has smallest (and odd) row index $s = 1$. Then, since the largest row index is 6, the Shift 12 Relation can be applied until the equation has column index $t < 18 = 12 + 6$. We are left with an equation of the form $-1 + \max\{\delta_{3,t}, \delta_{6,t}\} = \min\{\delta_{1,t}, \delta_{4,t}\}$ for some $2 \leq t \leq 17$. Again, all of these 16 equations do hold.

$$\begin{bmatrix}
0&1&1&2&2&3&3&2&4&1&3&2&2&3&3&4&4&5&5&4&6&3&5&4\\
1&0&2&1&1&2&2&1&3&2&2&3&3&2&4&3&3&4&4&3&5&4&4&5\\
1&2&0&1&1&2&2&3&3&2&4&1&3&2&2&3&3&4&4&5&5&4&6&3\\
2&1&3&0&2&1&1&2&2&3&3&4&4&3&5&2&4&3&3&4&4&5&5&4\\
2&3&1&2&0&1&1&2&2&3&3&2&4&1&3&2&2&3&3&4&4&5&5&4\\
1&2&2&1&3&0&2&1&1&2&2&3&3&4&4&3&5&2&4&3&3&4&4&5\\
1&2&2&3&1&2&0&1&1&2&2&3&3&2&4&1&3&2&2&3&3&4&4&5\\
2&3&1&2&2&1&3&0&2&1&1&2&2&3&3&4&4&3&5&2&4&3&3&4\\
2&1&3&2&2&3&1&2&0&1&1&2&2&3&3&2&4&1&3&2&2&3&3&4\\
3&4&2&3&1&2&2&1&3&0&2&1&1&2&2&3&3&4&4&3&5&2&4&3\\
3&2&4&1&3&2&2&3&1&2&0&1&1&2&2&3&3&2&4&1&3&2&2&3\\
2&3&3&4&2&3&1&2&2&1&3&0&2&1&1&2&2&3&3&4&4&3&5&2\\
2&3&3&2&4&1&3&2&2&3&1&2&0&1&1&2&2&3&3&2&4&1&3&2\\
3&2&4&3&3&4&2&3&1&2&2&1&3&0&2&1&1&2&2&3&3&4&4&3\\
3&4&2&3&3&2&4&1&3&2&2&3&1&2&0&1&1&2&2&3&3&2&4&1\\
4&3&5&2&4&3&3&4&2&3&1&2&2&1&3&0&2&1&1&2&2&3&3&4\\
4&5&3&4&2&3&3&2&4&1&3&2&2&3&1&2&0&1&1&2&2&3&3&2\\
3&4&4&3&5&2&4&3&3&4&2&3&1&2&2&1&3&0&2&1&1&2&2&3\\
3&4&4&5&3&4&2&3&3&2&4&1&3&2&2&3&1&2&0&1&1&2&2&3\\
4&5&3&4&4&3&5&2&4&3&3&4&2&3&1&2&2&1&3&0&2&1&1&2\\
4&3&5&4&4&5&3&4&2&3&3&2&4&1&3&2&2&3&1&2&0&1&1&2\\
5&6&4&5&3&4&4&3&5&2&4&3&3&4&2&3&1&2&2&1&3&0&2&1\\
5&4&4&3&5&4&4&5&3&4&2&3&3&2&4&1&3&2&2&3&1&2&0&1\\
4&5&5&6&4&5&3&4&4&3&5&2&4&3&3&4&2&3&1&2&2&1&3&0
\end{bmatrix}$$

Exponent matrix for Δ_{24}

4. THE X-MODULES

Recall that $J(i)$ denotes the ith row of the radical of Λ, that is, $J(i)$ is the radical of $P(i)$. Since Λ is Noetherian with Krull dimension 1, the global dimension of Λ is one more than the projective dimension of its radical. So, our task is reduced to finding the maximum projective dimension among the $J(i)$. As a first step in this section we shall see that the syzygies of $J(N-1)$ all decompose into direct sums of uniform modules, the X_k defined below. This decomposability makes tractable the computation of the projective dimensions of the X_k and, so, that of $J(N-1)$. We will denote the projective dimension of a Λ module by pd.

DEFINITION 4.1. For $k = 1, \dots, 3n-8$ we define the following uniform lattices X_k by means of their projective covers.

$$
\begin{aligned}
X_0 &= P(2),\\
X_k &= \pi^{-k}P(2k-1) + P(2k+2), \qquad 1 \le k \le n-2,\\
X_{n-1} &= \pi^{1-n}P(N-5) + P(N),\\
X_k &= \pi^{1-n}P(2N-2k-7) + \pi^{n-k-1}P(2N-2k),\\
&\qquad\qquad\qquad\qquad\qquad\qquad n \le k \le 2n-5,\\
X_{2n-4} &= \pi^{1-n}P(1) + \pi^{-n}P(5) + \pi^{3-n}P(8),\\
X_k &= \pi^{n-4-k}P(2k-2N+13) + \pi^{1-n}P(2k-2N+8),\\
&\qquad\qquad\qquad\qquad\qquad\qquad 2n-3 \le k \le 3n-9,\\
X_{3n-8} &= \pi^{1-n}P(N-8) + \pi^{4-N}P(N-3) + \pi^{3-n}P(N).
\end{aligned}
$$

Note that a quick inspection of the weighted link diagram of Λ shows that the links to $N-1$ are $N-8 \to_{n-2} N-1$, $N-3 \to_1 N-1$, and $N \to_n N-1$ where we have used (2.2) to compute the weights. Now Lemma 1.3 allows us to construct the projective cover of $J(N-1)$:

$$
\begin{aligned}
J(N-1) &= \pi^{n-2}P(N-8) + \pi^{1}P(N-3) + \pi^{n}P(N)\\
&= \pi^{N-3}X_{3n-8}.
\end{aligned}
$$

A similar computation show that $J(N) = X_{n-2}$.

PROPOSITION 4.2. (1) $\Omega X_{3n-8} = X_{3n-9} \oplus \pi^{3-n}X_{n-2}$,

(2) $\Omega X_{2n-4} = X_{2n-5} \oplus \pi^{3-n}X_2$,

(3) $\Omega X_k = X_{k-1}$ *for all other* $k \ge 1$.

Proof. An invocation of Lemma 1.10 will give us (1) once we show

$$\begin{aligned} X_{3n-9} &= \pi^{1-n}P(N-8) \cap \pi^{4-N}P(N-3), \\ \pi^{3-n}X_{n-2} &= \pi^{4-N}P(N-3) \cap \pi^{3-n}P(N), \\ \pi^{5-N}P(N-5) &= \pi^{1-n}P(N-8) \cap \pi^{3-n}P(N) \\ &= \pi^{1-n}P(N-8) \cap \pi^{4-N}P(N-3) \cap \pi^{3-n}P(N). \end{aligned}$$

Similarly for (2) we need

$$\begin{aligned} X_{2n-5} &= \pi^{1-n}P(1) \cap \pi^{3-n}P(8), \\ \pi^{3-n}X_2 &= \pi^{-n}P(5) \cap \pi^{3-n}P(8), \\ \pi^{1-n}P(3) + \pi^{4-n}P(8) &= \pi^{1-n}P(1) \cap \pi^{-n}P(5) \\ &= \pi^{1-n}P(1) \cap \pi^{-n}P(5) \cap \pi^{3-n}P(8). \end{aligned}$$

The above eight equalities along with those from (3) are proved as described in Section 3. ∎

An immediate consequence of this proposition is that, for every k from 1 to $3n-8$, X_k has projective dimension k. In particular, since $J(N-1) \cong X_{3n-8}$, we have the following.

COROLLARY 4.3. *The projective dimension of* $J(N-1)$ *is* $3n-8$.

5. THE Y MODULES

There is another class of uniform modules we use to compute the global dimension of Λ.

DEFINITION 5.1. For $0 \le k \le N-8$ define:

$$\begin{aligned} Y_0 &= P(1) + \pi^2 P(8), \\ Y_k &= P(k) + \pi^2 P(k+8) \qquad \text{for even } k \ge 2, \\ Y_k &= P(k) + \pi^{-2} P(k+8) \qquad \text{for odd } k. \end{aligned}$$

We list the syzygies of these Y modules. The proof of the following proposition once again follows the outline in Section 3 and is omitted.

PROPOSITION 5.2. (1) $\Omega Y_0 = \pi^{n-1}X_{N-5}$,

(2) $\Omega Y_k = \pi^x Y_{k+1}$, $x = (k+1)/2$, *for odd* k,

(3) $\Omega Y_2 = \pi^2 X_3$,

(4) $\Omega Y_k = \pi^x P(k+3)$, $x = -k/2$, *for even* $k \ge 4$.

Coupling the above proposition with Proposition 4.2, we can list the projective dimensions of all of the Y modules.

COROLLARY 5.3.

$$\operatorname{pd}(Y_k) = \begin{cases} N-4, & \textit{if } k=0, \\ 5, & \textit{if } k=1, \\ 4, & \textit{if } k=2, \\ 2, & \textit{if } 3 \le k \textit{ is odd}, \\ 1, & \textit{if } 4 \le k \textit{ is even}. \end{cases}$$

6. THE GLOBAL DIMENSION OF Λ_N IS $2N-8$

In this section we bring together the two types of uniform modules, the X and Y, to compute the projective dimensions of the simple right modules $S_i = P(i)/J(i)$. The statement that the global dimension is $2N-8$ is an immediate corollary of the following theorem. Note in particular that the unique simple right module with maximal projective dimension is given by $\operatorname{pd}(S_1) = 2N-8$.

THEOREM 6.1. (1) $\operatorname{pd}(S_N) = n-1$,

(2) $\operatorname{pd}(S_{N-2}) = n$,

(3) $\operatorname{pd}(S_i) = 2N-8-i/2$ *for* i *even*, $2 \le i \le N-4$,

(4) $\operatorname{pd}(S_i) = 2N-8-(i-1)/2$ *for* i *odd*, $1 \le i \le N-1$.

Proof. Since the ith simple module S_i has first syzygy $J(i)$ and $\operatorname{pd}(X_k) = k$, formula (1) of Theorem 6.1 follows directly from $J(N) = X_{n-2}$. Also, formula (4) with $i = N-1$ follows from $J(N-1) = X_{3n-8}$.

To help verify the remaining formulas, we present a series of torsion modules T_i, for $1 \le i \le N-2$, each presented as either $P(i)$ or $J(i+2)$ modulo a submodule of the form π^x times an X or Y from Sections 4 and 5. For each i we let C_i denote the set of indices of simple modules occurring as composition factors of T_i. The simple modules corresponding to indices explicitly listed as members of C_i occur as composition factors of T_i with multiplicity 1.

$$T_{N-2} = J(N)/\pi X_{n-1} \qquad \text{with } C_{N-2} = \{N-2\},$$

$$T_{N-4} = J(N-2)/\pi^2 X_n \qquad \text{with } C_{N-4} = \{N-4, N-1\},$$

$$T_i = J(i+2)/\pi^x Y_{i-3},$$

$$x = 1 - i/2, \text{ for } i \text{ even and } N-6 \ge i \ge 6 \text{ with } C_i = \{i, i+2, i+4\},$$

$$T_4 = P(4)/\pi^{n-1}X_{N-3} \qquad \text{with } C_4 = \{4, 6\},$$

$$T_2 = P(2)/\pi^{n-1}X_{N-4} \qquad \text{with } C_2 = \{2, 4\},$$

$$T_{N-3} = J(N-1)/\pi^{n-2}Y_{N-8} \qquad \text{with } C_{N-3} = \{N-3, N-1\},$$

$$T_i = J(i+2)/\pi^x Y_{i-5},$$

$x = (i-1)/2$, for i odd and $N-5 \geq i \geq 5$ with $C_i \supseteq \{i, i+2, i+4\}$ and all other composition factors of T_i have indices larger than $i+4$,

$$T_3 = J(5)/\pi^3 Y_6 \qquad \text{with } C_3 = \{3, 5, 7\},$$

$$T_1 = J(3)/\pi^2 Y_4 \qquad \text{with } C_1 = \{1, 3, 5\}.$$

The verification that the T_i's have composition factors as claimed is done by translation to equations in the λ_{ij}, just as the proofs of Sections 4 and 5 are done. The only differences come from the appearance of $J(i)$ in place of $P(i)$ and the occurrence of the composition factors indicated by the C_i. An example will suffice to illustrate the differences and similarities, so consider T_1: Let y_j denote the exponent of Y_4 in the jth column. The inclusion $\pi^2 Y_4 \subseteq J(3)$ is equivalent to $2 + y_j \geq \lambda_{3j}$ for $j \neq 3$ and $2 + y_3 \geq 1$. The statement that $C_1 = \{1, 3, 5\}$ amounts to saying all these inequalities are actually equalities except for $j = 1, 3, 5$ and in these cases (since no simple module occurs as a composition factor with multiplicity more than 1) we have $2 + y_1 = \lambda_{31} + 1$, $2 + y_3 = 1 + 1$, and $2 + y_5 = \lambda_{35} + 1$. As in Sections 4 and 5 we omit further details of the proofs.

We return once again to considering Theorem 6.1.

Note that formula (2) of Theorem 6.1 is an immediate consequence of the isomorphism $T_{N-2} \cong S_{N-2}$: there is then a short exact sequence

$$0 \to X_{n-1} \to J(N) \to S_{N-2} \to 0,$$

where $\mathrm{pd}(J(N)) = n-2$ and $\mathrm{pd}(X_{n-1}) = n-1$. Consequently $\mathrm{pd}(S_{N-2}) = 1 + (n-1)$.

Formula (3) of Theorem 6.1 for $i = N-4$ follows from the existence of T_{N-4}. Since $T_{N-4} = J(N-2)/\pi^2 X_n$ where $\mathrm{pd}(J(N-2)) = n-1$ and $\mathrm{pd}(X_n) = n$, we see that $\mathrm{pd}(T_{N-4}) = n+1$. Now the composition factors of T_{N-4} are S_{N-4} and S_{N-1}, each occurring once. There is a path in the link diagram from $N-1$ to $N-2$ of weight $2 - n = \lambda_{N-2, N-1} = \rho^1_{N-2, N-1}$ and length 2, and every path from $N-1$ to $N-2$ of length greater than 2 has weight greater than $2-n$. Applying Corollaries 1.6 and 1.7 shows that S_{N-1} is a composition factor of $T_{N-4}J$, that is, S_{N-1} is the socle of T_{N-4}. We have a short exact sequence

$$0 \to S_{N-1} \to T_{N-4} \to S_{N-4} \to 0.$$

Since $\mathrm{pd}(S_{N-1}) = 3n-8$ and $\mathrm{pd}(T_{N-4}) = n+1$, we have $\mathrm{pd}(S_{N-4}) = 1 + (3n-7)$ as desired.

The remaining formulas in Theorem 6.1 are proved by "downward" induction: we assume the result for simple modules $S(k)$ with $N \geq k > i$ and show that $\mathrm{pd}(S_i) = 1 + \mathrm{pd}(S_{i+2})$. The formula is then seen to hold for S_i. The inductive basis consists of the already established formulas $\mathrm{pd}(S_N) = n - 1$, $\mathrm{pd}(S_{N-1}) = 3n - 7$, $\mathrm{pd}(S_{N-2}) = n$, and $\mathrm{pd}(S_{N-4}) = 3n - 6$.

The details of the inductive step vary with i, but the overall outline is the same for all i except $i = 4, 2$. We handle these anomalous cases first. Here T_i has length 2 with composition factors S_i and S_{i+2}. The presentation $T_i = P(i)/X$ shows that T_i is uniserial with socle S_{i+2}. Also $\mathrm{pd}(T_i) = 1 + \mathrm{pd}(X) < N < \mathrm{pd}(S_{i+2})$ by the inductive hypothesis. We have the short exact sequence

$$0 \to S_{i+2} \to T_i \to S_i \to 0.$$

So $\mathrm{pd}(S_i) = 1 + \mathrm{pd}(S_{i+2})$.

Now we deal with the remaining i: even i with $6 \leq i \leq N - 6$ and odd i with $1 \leq i \leq N - 3$. We first compute the projective dimension of T_i. We have a short exact sequence

$$0 \to Y \to J(i+2) \to T_i \to 0,$$

where Y is one of the modules from Section 5. It is always the case that $\mathrm{pd}(Y) < \mathrm{pd}(J(i+2))$, so we have $\mathrm{pd}(T_i) = \mathrm{pd}(J(i+2))$.

We observe that $i + 2 \in C_i$ for all i under consideration. Examining the remaining indices in C_i and using the inductive hypothesis, we find that $\mathrm{pd}(S_j) < \mathrm{pd}(S_{i+2}) - 1$ for any $j \in C_i - \{i, i+2, i+4\}$.

Now there are links $i \to i + 2$ and $i + 4 \to i + 2$, so Lemma 1.2 with Corollaries 1.6 and 1.7 shows that S_i and S_{i+4} (if it is a composition factor of T_i) are summands of T_i/T_iJ, and of no other Loewy layer of T_i. Also observe that the link diagram has paths from $i + 2$ to $i + 2$ of weight $1 = \rho^1_{i+2,i+2}$ and length 2, while any longer path has greater weight. Again Corollaries 1.6 and 1.7 can be applied, and we see that S_{i+2} is a summand of T_iJ/T_iJ^2 and no other Loewy layer of T_i.

In summary, every composition factor of T_iJ other than S_{i+2}, and every composition factor of T_iJ^2, has projective dimension at least 2 less than $\mathrm{pd}(S_{i+2})$.

Consider the short exact sequences

$$0 \to T_iJ^2 \to T_iJ \to T_iJ/T_iJ^2 \to 0$$

and

$$0 \to T_iJ \to T_i \mapsto T_i/T_iJ \to 0.$$

In the first sequence we know that $\mathrm{pd}(T_i J^2) \leq \mathrm{pd}(S_{i+2}) - 2$ since all the composition factors of $T_i J^2$ have projective dimension this small. Also S_{i+2} is a summand of the semisimple module $T_i J / T_i J^2$ (and all other summands have smaller projective dimension) so $\mathrm{pd}(T_i J / T_i J^2) = \mathrm{pd}(S_{i+2})$. Putting these observations together gives us $\mathrm{pd}(T_i J) = \mathrm{pd}(S_{i+2})$.

In the second sequence we now have $\mathrm{pd}(T_i J) = \mathrm{pd}(S_{i+2}) > \mathrm{pd}(J(i+2)) = \mathrm{pd}(T_i)$, so $\mathrm{pd}(T_i / T_i J) = 1 + \mathrm{pd}(S_{i+2})$. But $T_i / T_i J$ is semisimple with all summands other than S_i having projective dimension smaller than $\mathrm{pd}(S_{i+2}) < \mathrm{pd}(T_i / T_i J)$. We must have $\mathrm{pd}(S_i) = \mathrm{pd}(T_i / T_i J) = 1 + \mathrm{pd}(S_{i+2})$. ∎

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